February 1995 PAR–LPTHE 95/05

RENORMALIZATION GROUP SOLUTION FOR THE TWO-DIMENSIONAL RANDOM BOND POTTS MODEL WITH BROKEN REPLICA SYMMETRY

Vik.S.Dotsenko

Landau Institute for Theoretical Physics,

Russian Academy of Sciences,

Kosygina 2, 117940 Moscow, Russia

Vl.S.Dotsenko*, **M. Picco and P.Pujol**

LPTHE†

Université Pierre et Marie Curie, PARIS VI

Université Denis Diderot, PARIS VII

Boite 126, Tour 16, 1er étage

4 place Jussieu

F-75252 Paris CEDEX 05, FRANCE

ABSTRACT

We find a new solution of the renormalization group for the Potts model with ferromagnetic random valued coupling constants. The solution exhibits universality and broken replica symmetry. It is argued that the model reaches this universality class if the replica symmetry is broken initially. Otherwise the model stays with the replica symmetric renormalization group flow and reaches the fixed point which has been considered before.

*Also at the Landau Institute for Theoretical Physics, Moscow

†Laboratoire associé No. 280 au CNRS

The problem of new critical behavior induced by randomness in spin systems has a considerable history. Starting with a classical φ^4 problem, the modified critical behavior has later been studied for the two-dimensional Ising and Potts models by various renormalization group techniques, and by numerical simulations. Incomplete list of references is provided in [1-12]. Replicas has been used generally to deal with the quenched disorder and replica symmetric solutions have generally been looked for. The first example of replica symmetry broken solutions of the renormalization group has been suggested in [13], in the context of the φ^4 model. In this letter we report on the replica symmetry broken solution for the two-dimensional Potts model with random bonds. The model reaches this solution if the replica symmetry is broken initially. In contrast, the two-dimensional Ising model turns out to be stable with respect to replica symmetry breaking [4]. It reaches always the replica symmetric critical behavior which has been studied earlier [5-12].

For theoretical study one uses models with a weak disorder, e.g. models with spin couplings having small fluctuations around a mean ferromagnetic value. This gives a possibility to study the model in continuum, because one reaches the critical point sufficiently close before the randomness becomes important. For the two-dimensional Potts model in particular this allows to use eventually the renormalization group based on the conformal theory of the unperturbed model. In this approach the effective theory could be described by the Hamiltonian

$$H = H_0 + \int d^2x\, m(x)\varepsilon(x) \tag{1}$$

where H_0 represents, symbolically, the conformal theory of the unperturbed model, while the second term with a spatially random mass $m(x)$ coupled to the energy operator represents the effective randomness due to spatially inhomogeneous coupling constants of spins. Replicating the model and taking the average of the partition function over $m(x)$ one gets the effective homogeneous theory with the Hamiltonian:

$$H = \sum_{a=1}^{n} H_0^a + g\int d^2x \sum_{a\neq b} \varepsilon_a(x)\varepsilon_b(x) \tag{2}$$

where g is defined by

$$\langle m(x)m(x')\rangle = g\delta(x-x') \tag{3}$$

Without loss of generality one could assume a Gaussian distribution for $m(x)$ because the terms in the effective Hamiltonian produced by higher moments are irrelevant, in the sense of the renormalization group. Also, a spatially uncorrelated distribution of coupling constants, and so of $m(x)$, is being assumed. For extra details on the definition of this model the reader could consult the papers [8,12].

The β - function of the renormalization group for the model (2) has been derived, up to second order of perturbation theory (third order in g), in [8], with the following result:

$$\frac{dg}{d\xi} = \beta(g) \tag{4}$$

$$\beta(g) = -3\epsilon g + 4\pi(n-2)g^2 - 16\pi^2(n-2)g^3 + O(g^4) \tag{5}$$

Here ξ is the renormalization group log-scale parameter, and ϵ is related to the central charge of the conformal theory for the Potts model at the critical point, on a homogeneous lattice. The following parametrization is being used [12]:

$$c = 1 - 24\alpha_0^2 = 1 - 6(\alpha_+ + \alpha_-)^2 \tag{6}$$

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}, \quad \alpha_+\alpha_- = -1 \tag{7}$$

$$\alpha_+^2 = \frac{4}{3} + \epsilon \tag{8}$$

The case of $\alpha_+^2 = \frac{4}{3}$, $c = \frac{1}{2}$ corresponds to the conformal theory for the Ising model. By shifting α_+^2 one shifts the central charge c. In particular, for the 3 - components Potts model, the parameters of the associated conformal theory are :

$$c = \frac{4}{5}, \quad \alpha_+^2 = \frac{6}{5} \tag{9}$$

which corresponds to

$$\epsilon = -\frac{2}{15} \tag{10}$$

The energy operator ε of the Potts model corresponds, in the conformal theory classification, to the operator $\Phi_{1,2}$[14]. In general, the dimensions of the conformal operators $\Phi_{n',n}$ are given by the Kac formula [15]:

$$\Delta_{n',n} = \alpha_{n',n}^2 - 2\alpha_0\alpha_{n',n} = \frac{(\alpha_- n' + \alpha_+ n)^2 - (\alpha_+ + \alpha_-)^2}{4} \tag{11}$$

$$\alpha_{n',n} = \frac{1-n'}{2}\alpha_- + \frac{1-n}{2}\alpha_+ \tag{12}$$

For $\varepsilon \sim \Phi_{1,2}$ one gets

$$\alpha_{1,2} = -\frac{\alpha_+}{2} \tag{13}$$

$$\Delta_\varepsilon = 2\Delta_{1,2} = 2(\alpha_{1,2}^2 - 2\alpha_0\alpha_{1,2}) = \frac{3}{2}\alpha_+^2 - 1 = 1 + \frac{3}{2}\epsilon \tag{14}$$

So, for the case of the Ising model, $\epsilon = 0$, $\Delta_\varepsilon = 1$, the perturbation in (2) is marginal, and then one defines the renormalization group for the Potts model in terms of the ϵ - expansion technique [8].

The renormalization of the operator $\varepsilon(x)$ has also been found in [8], up to the second order, with the following result:

$$\varepsilon(x) \to \varepsilon_{ren}(x) = Z_\varepsilon(\xi)\varepsilon(x) \tag{15}$$

$$\frac{d\log Z_\varepsilon}{d\xi} = \gamma_\varepsilon(g) \tag{16}$$

$$\gamma_\varepsilon(g) = 4\pi(n-1)g - 8\pi^2(n-1)g^2 + O(g^3) \tag{17}$$

Finally, the renormalization of the spin operator has been found in [12], up to the third order:

$$\sigma(x) \to \sigma_{ren}(x) = Z_\sigma(\xi)\sigma(x) \tag{18}$$

$$\frac{d\log Z_\sigma}{d\xi} = \gamma_\sigma(g) \tag{19}$$

$$\begin{aligned}\gamma_\sigma(g) = -3(n-1)\pi^2\epsilon\left(1 + 2\frac{\Gamma^2(-\frac{2}{3})\Gamma^2(\frac{1}{6})}{\Gamma^2(-\frac{1}{3})\Gamma^2(-\frac{1}{6})}\right)g^2 \\ +4(n-1)(n-2)\pi^3 g^3 + O(g^4)\end{aligned} \tag{20}$$

Here $\Gamma(z)$ is the Euler Γ-function.

Using results for $\beta, \gamma_\varepsilon$ and γ_σ, eqs. (5),(17),(20), where one puts eventually $n = 0$, the following results have been obtained for g_c, Δ_ε [8] and Δ_σ [12]:

$$g_c = -\frac{3}{8\pi}\epsilon + \frac{9}{16\pi}\epsilon^2 + O(\epsilon^3) \tag{21}$$

$$\Delta'_\varepsilon = \Delta_\varepsilon - \gamma_\varepsilon(g_c) = \Delta_\varepsilon - \frac{3}{2}\epsilon + \frac{9}{8}\epsilon^2 + O(\epsilon^3) = 1 + \frac{9}{8}\epsilon^2 + O(\epsilon^3) \tag{22}$$

$$\Delta'_\sigma = \Delta_\sigma - \gamma_\sigma(g_c) = \Delta_\sigma - \frac{27}{32}\frac{\Gamma^2(-\frac{2}{3})\Gamma^2(\frac{1}{6})}{\Gamma^2(-\frac{1}{3})\Gamma^2(-\frac{1}{6})}\epsilon^3 + O(\epsilon^4) \tag{23}$$

For the 3-component Potts model, $\epsilon = -\frac{2}{15}$, one gets the following numerical values:

$$\Delta'_\epsilon = 1,02 + O(\epsilon^3) \tag{24}$$

$$\Delta'_\sigma = \frac{2}{15} + 0,00132 + O(\epsilon^4) = 0,13465 + O(\epsilon^4) \tag{25}$$

In the solution just outlined one assumes that the replica symmetry is not broken initially, and then it is preserved by the renormalization group. Physical arguments for a more general approach, namely to start with a Hamiltonian in which the replica symmetry is lifted, have been suggested in [13]. For the case of the Potts model this amounts to replacing the effective Hamiltonian (2) with:

$$H = \sum_{a=1}^{n} H_0^a + \int d^2x \sum_{a\neq b} g_{ab}\varepsilon_a(x)\varepsilon_b(x) \tag{26}$$

where g_{ab} is a Parisi type matrix [16]. We shall assume next this conjecture and give the solution of the theory (26), in order to allow, on the basis of the results obtained, to verify the conjecture itself.

By using the techniques of [12] the generalization of the renormalization group equations is straightforward. Eqs. (4) and (5) are replaced by

$$\frac{dg_{ab}}{d\xi} = \beta_{ab} \tag{27}$$

$$\beta_{ab} = -3\epsilon g_{ab} + 4\pi(g^2)_{ab} - 16\pi^2((g^2)_{aa}g_{ab} - (g_{ab})^3) \tag{28}$$

Here $(g^2)_{ab} = \sum_c g_{ac}g_{cb}$. The fixed point matrix g_{ab} should satisfy the equation:

$$-3\epsilon g_{ab} + 4\pi(g^2)_{ab} - 16\pi^2((g^2)_{aa}g_{ab} - (g_{ab})^3) = 0 \tag{29}$$

We always assume the diagonal elements of the coupling matrix g_{ab} to be zero. This is because the corresponding terms could always be absorbed into $\sum_{a=1}^n H_0^a$, in eq.(26). Then, for the Parisi type matrices, one has in general the following rules [16]:

$$g_{ab} \to g(x) \tag{30}$$

$$(g^2)_{ab} \rightarrow -2g(x)\int_0^1 dyg(y) - \int_0^x dy(g(x)-g(y))^2 \tag{31}$$

$$(g^2)_{aa} \rightarrow -\int_0^1 dyg^2(y) \tag{32}$$

Here x is the continuous parameter which replaces the matrix indices: $g_{ab} \sim g(a-b) \sim g(x)$. Putting $\tau = 3|\epsilon|$ (ϵ is assumed to be negative in its definition (8)), replacing $g \rightarrow \frac{1}{4\pi}g$, and using the prescriptions (30) - (32), one gets from (29) the following equation for $g(x)$:

$$\tau g(x) - 2\bar{g}g(x) - \int_0^x dy(g(x)-g(y))^2 + g^3(x) + g(x)\bar{g^2} = 0 \tag{33}$$

Here $\bar{g} = \int_0^1 dyg(y)$, $\bar{g^2} = \int_0^1 dyg^2(y)$. Note that the structure of the fixed-point equation (33) coincides with the saddle-point equation for the Parisi order parameter function in the infinite-range spin-glasses near the phase transition point (the parameter τ in eq.(33) corresponds to the reduced temperature $\tau = (1-T/T_c) << 1$ in the spin-glass model)[16]. The solution of this equation is straightforward. Taking a derivative with respect to x one gets:

$$\tau g'(x) - 2\bar{g}g'(x) - 2g'(x)\int_0^x dy(g(x)-g(y)) + 3g'(x)g^2(x) + g'(x)\bar{g^2} = 0 \tag{34}$$

So, either

$$g'(x) = 0 \tag{35}$$

or

$$\tau - 2\bar{g} - 2\int_0^x dy(g(x)-g(y)) + 3g^2(x) + \bar{g}^2 = 0 \tag{36}$$

Differentiating again one gets

$$-2g'(x)x + 6g'(x)g(x) = 0 \tag{37}$$

There are two solutions:

$$g(x) = const \equiv g_1 \tag{38}$$

and

$$g(x) = \frac{1}{3}x \tag{39}$$

Next, we take $g(x)$ in the form:

$$g(x) = \begin{cases} \frac{1}{3}x, & 0 < x < x_1 \\ g_1, & x_1 < x < 1 \end{cases} \tag{40}$$

with

$$x_1 = 3g_1 \tag{41}$$

Then, we put back $g(x)$ into the original equation (33). In particular

$$\bar{g} = g_1 - \frac{3}{2}g_1^2, \quad \bar{g^2} = g_1^2 - 2g_1^3 \tag{42}$$

Substituting (40),(42) into (33), either for $0 < x < x_1$ or for $x_1 < x < 1$ one gets, after some simple algebra

$$g_1 \approx \frac{1}{2}\tau + \frac{1}{2}\tau^2, \quad x_1 \approx \frac{3}{2}\tau + \frac{3}{2}\tau^2 \tag{43}$$

up to the second order in $\tau = 3|\epsilon|$. This solution can be compared with the replica symmetry one,

$$g_{ab} \sim g(x) = const = \frac{1}{2}\tau + \frac{1}{4}\tau^2, \; 0 < x < 1 \tag{44}$$

given by eq.(21) (after a rescaling $g_{ab} \to \frac{1}{4\pi}g_{ab}$).

Note that in terms of the original "dynamical" eqs.(27),(28) the *continuous* fixed-point solution, eq.(40), is the only one which is (marginally) attractive. This is guaranteed by the fact that the eigenvalues spectrum of the corresponding Hessian of the infinite-range spin-glass problem is known to be non-positive. On the other hand, all the other non-trivial fixed-point solutions which have step-like structure (they correspond to finite number of RSB steps in the replica matrix g_{ab}) are unstable because for any *finite* number of steps there exist positive finite eigenvalues [16].

We can now find the dimensions of the operators ε and σ, for the solution (40). Again, with a straightforward generalization, one finds for $\gamma_\varepsilon(g)$ and $\gamma_\sigma(g)$ the following expressions:

$$\gamma_\varepsilon(g) = 4\pi\frac{1}{n}\sum_{ab} g_{ab} - 8\pi^2(g^2)_{aa} + O(g^3) \tag{45}$$

$$\gamma_\sigma(g) = 3\pi^2\epsilon\left(1 + 2\frac{\Gamma^2(-\frac{2}{3})\Gamma^2(\frac{1}{6})}{\Gamma^2(-\frac{1}{3})\Gamma^2(-\frac{1}{6})}\right)(g^2)_{aa} + 8\pi^3(g^3)_{aa} + O(g^4) \tag{46}$$

Using the Parisi ansatz $g_{ab} \to g(x)$ with the rules (30)-(32) and, in addition,

$$(g^3)_{aa} \to \int_0^1 dx(xg^3(x) + 3g(x)\int_0^x dy\, g^2(y)) \tag{47}$$

one obtains the following expressions:

$$\gamma_\varepsilon(g) = -4\pi \int_0^1 g(x)dx + 8\pi^2 \int_0^1 g^2(x)dx + O(g^3) \tag{48}$$

$$\gamma_\sigma(g) = -3\pi^2\epsilon\left(1 + 2\frac{\Gamma^2(-\frac{2}{3})\Gamma^2(\frac{1}{6})}{\Gamma^2(-\frac{1}{3})\Gamma^2(-\frac{1}{6})}\right)\int_0^1 g^2(x)dx$$
$$+ 8\pi^3 \int_0^1 \left(xg^3(x) + 3g(x)\int_0^x g^2(y)dy\right)dx + O(g^4) \tag{49}$$

Simple analysis of these expressions shows that the modification of Δ'_σ for the solution (40) for $g(x)$ will be of order ϵ^4. As we haven't kept the $\sim \epsilon^4$ terms in $\gamma_\sigma(g)$ the accuracy is not sufficient to use the modified $g(x)$. To ϵ^3 order, Δ'_σ remains the same, eqs.(23),(25). On the other hand, the accuracy is sufficient for $\gamma_\varepsilon(g)$ which depends of terms of order $\sim g$ and $\sim g^2$ with coefficients of order ~ 1. Simple calculation with $\gamma_\varepsilon(g)$ in (45) leads to

$$\Delta''_\varepsilon = \Delta_\varepsilon - \gamma_\varepsilon(g) = \Delta_\varepsilon - \frac{3}{2}\epsilon + O(\epsilon^3) = 1 + O(\epsilon^3) \tag{50}$$

which can be compared to the solution of Δ'_ε given by eq.(24): $\Delta'_\epsilon = 1,02 + O(\epsilon^3)$

To conclude, we have found an explicit form, at the order ϵ^2, of replica symmetry broken fixed point in the 3-states Potts model with random ferromagnetic bonds. We have also calculated one observable quantity, the dimension of the energy operator Δ''_ε, which could distinguish this universality class in a numerical experiment [17].

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